Finite Math - J-term 2017

Lecture Notes - 1/4/2017

Homework

- Section 3.1 - 9, 11, 15, 18, 20, 22, 24, 26, 34, 50, 55, 58, 71, 80, 81

Section 2.6 - Logarithmic Functions

Example 1. *The isotope carbon-14 has a half-life (the time it takes for the isotope to decay to half of its original mass) of* 5730 *years.*

(a) *At what rate does carbon-*14 *decay?*

(b) *How long would it take for* 90% *of a chunk of carbon-*14 *to decay?*

Solution.

(a) *Suppose we have an initial mass of* M_0. *After half of it decays, the mass will be* $\frac{M_0}{2}$ *and this happens after* $t = 5730$ *years has elapsed. Plugging all this into our model, we get*

$$\frac{M_0}{2} = M_0 e^{r(5730)} \iff \frac{1}{2} = e^{5730r}$$

Applying the natural log to each side gives

$$\ln \frac{1}{2} = \ln e^{5730r}$$

Using properties of logarithms, we have

$$\ln \frac{1}{2} = \ln 2^{-1} = -\ln 2$$

and

$$\ln e^{5730r} = 5730r \ln e = 5730r$$

so that

$$-\ln 2 = 5730r.$$

Solving for r, *we get*

$$r = -\frac{\ln 2}{5730} \approx -0.00012$$

*This means that carbon-*14 *decays at a rate of* 0.12% *per year.*

(b) *If the mass of* M_0 *loses* 90% *of its mass, we're looking for the time it takes for only* $0.1M_0$ *to remain. So,*

$$0.1M_0 = M_0 e^{-0.00012t}$$

and canceling the M_0*'s gives*

$$0.1 = e^{-0.00012t}.$$

Hit both sides of this with $\ln$ *to get*

$$\ln 0.1 = \ln e^{-0.00012t} = -0.00012t.$$

Solve for t

$$t = -\frac{\ln 0.1}{0.00012} \approx 19,188.21.$$

So, it would take about $12,188.21$ *years for* 90% *of the original mass to decay.*

Section 3.1 - Simple Interest

Suppose you make a deposit or investment of P dollars or you take out a loan of P dollars. The amount P is called the *principal*.

All of these things have an *interest rate* attached to them, essentially rent on the money, which is paid as *interest*.

Simple Interest. Simple interest is computed as

$$I = Prt$$

where I = interest, P = principal, r = annual simple interest rate (written as a decimal), and t = time in years.

Example 2. *Suppose you deposit* $\$2,000$ *into a savings account with an annual simple interest rate of* 6%*. How much interest will accrue after* 6 *months?*

Solution. 6 *months is* 0.5 *years, so* $t = 0.5$*. The interest is* 6%*, so* $r = 0.06$*. The principal is* $P = 2000$*. Plug all this is to get*

$$I = 2000(0.06)(0.5) = 60.$$

So, $\$60$ *would have accrued after* 6 *months.*

Future Value. Often, we might be more curious about how much will be in the account or how much will be owed on the loan after a certain period. This amount is called the *future value*. Another name for principal is *present value*. It is found by simply adding the original investment/loan amount to the interest accrued.

Definition 1 (Future Value)**.**

$$A = P + I = P + Prt$$

and in a simplified form

$$A = P(1 + rt)$$

where A *=future value,* P *=principal/present value,* r = *annual simple interest rate,* t = *time in years.*

Example 3. *Suppose you take out a* $10,000 *loan at a simple annual interest rate of* 3.2%*. How much would be due on the loan after* 10 *months?*

Solution. *Principal* $P = 10000$
interest rate $r = 0.032$
10 *months* $= \frac{10}{12}$ *years* $= \frac{5}{6}$ *years, so* $t = \frac{5}{6}$ *The future value is then*

$$\begin{aligned} A &= 10000\left(1 + (0.032)\left(\frac{5}{6}\right)\right) \\ &\approx 10000(1.027) = \$10,266.67 \end{aligned}$$

Example 4. *You make an investment of* $3,000 *at an annual rate of* 4.5%*. What will be the value of your investment after* 30 *days? (Assume there are* 360 *days in a year.)*

Solution. $3,011.25

We can also use this formula to predict what interest rate we need or how much principal to take out/deposit.

Example 5. *You're looking to invest* $5,000 *and make* $1,000 *in interest after* 2 *years. What annual rate on your investment will you need to accomplish this?*

Solution. $P = 5000$ *and* $I = 1000$*, so the future value is* $A = 6000$*. The time is* $t = 2$*, so plugging all this into the formula gives*

$$6000 = 5000(1 + r(2))$$

and we need to solve for r*.*

$$\begin{aligned} 6000 &= 5000(1 + 2r) \\ &= 5000 + 10000r \\ \implies 1000 &= 10000r \\ \implies r &= \tfrac{1000}{10000} = 0.1 \end{aligned}$$

So we would need an annual rate of 10% *to make* $\$1,000$ *in interest after* 2 *years. (We actually could have just used the formula for interest,* $I = Prt$*, to solve this problem.)*

Example 6. *You invest* $\$4,000$ *at an annual rate of* 3.9%*. How long will it take for the investment to be worth* $\$5,000$*? Give your answer in years, correct to* 2 *decimal places.*

Solution. 6.41 *years*

One often uses a brokerage firm when making investments, many of which charge you a fee based on the transaction amount (principle) when both buying AND selling stocks.

Example 7. *Suppose a brokerage firm uses the following commission schedule*

Principal	*Commission*
Under $3,000	*$25+1.8% of principal*
$3,000 - $10,000	*$37+1.4% of principal*
Over $10,000	*$107+0.7% of principal*

An investor purchases 450 *shares of a stock at* $\$21.40$ *per share, keeps the stock for* 26 *weeks, then sells the stock for* $\$24.60$ *per share. What was the annual interest rate earned on the investment?*

Solution. *To purchase* 450 *shares will cost* $\$21.40(450) = \$9,630$*. This falls into the second fee range of the commission schedule, so the transaction fee will be*

$$\$37 + 0.014(\$9630) = \$171.82.$$

Thus, the total initial investment is

$$\$9,630 + \$171.82 = \$9,801.82.$$

Next, the investor sells the stock for

$$\$24.60(450) = \$11,070$$

This falls into the third fee range on the schedule, so the commission is

$$\$107 + 0.007(\$11,070) = \$184.49.$$

Thus, the net return on the investment is

$$\$11,070 - \$184.49 = \$10,885.51.$$

Now, using the total investment as the principal and the net return as the future value, we can use the future value formula to figure out the annual interest rate *earned.* $P = 9801.82$, $A = 10885.51$, *the time elapsed was* 26 *weeks, and there are* 52 *weeks in a year, so* $t = \frac{26}{52} = 0.5$.

$$\begin{aligned} 10885.51 &= 9801.82(1 + 0.5r) \\ &= 9801.82 + 4900.91r \\ \implies 1083.69 &= 4900.91r \\ \implies r &= \frac{1083.69}{4900.91} \approx 0.22112 \end{aligned}$$

So the interest rate earned was 22.112%.

Example 8. *Suppose a brokerage firm uses the following commission schedule*

Principal	*Commission*
Under $3,000	*$32+1.8% of principal*
$3,000 - $10,000	*$56+1% of principal*
Over $10,000	*$106+0.5% of principal*

An investor purchases 75 *shares of a stock at* \$37.90 *per share, keeps the stock for* 150 *days, then sells the stock for* \$41.20 *per share. What was the annual interest rate earned on the investment? (Again, assume a* 360*-day year.)*

Solution. 6.352%

Average Daily Balance. A common method for calculating interest on a credit card is to use the *average daily balance method*. As the name suggests, the average daily balance is computed, then the interest is computed on that.

Example 9. *A credit card has an annual interest rate of* 19.99% *and interest is calculated using the average daily balance method. If the starting balance of a* 30*-day billing cycle is* \$523.18 *and purchases of* \$147.98 *and* \$36.27 *are posted on days* 12 *and* 25*, respectively, and a payment of* \$200 *is credited on day* 17*, what will be the balance on the card at the start of the next billing cycle?*

Solution. *We must figure out what the balance is on each day of the month. At the end of day* 1*, the balance is* \$523.18. *The first transaction happens on day* 12*, which is a purchase of* \$147.98*, making the balance* \$671.16. *The next transaction is on day* 17*, a payment of* \$200*, making the balance* \$471.16. *The next, and final, transaction is on day* 25 *which is a purchase of* \$36.27*, making the balance* \$507.43. *It helps to make a chart of this data*

Day 1-11:	\$523.18	*(11 days)*
Day 12-16:	\$671.16	*(5 days)*
Day 17-24:	\$471.16	*(8 days)*
Day 25-30:	\$507.43	*(6 days)*

To find the average daily balance, we can take the sum of the balance at the end of each day, then divide by the number of days.

$$SUM = 11(523.18) + 5(671.61) + 8(471.16) + 6(507.43) = \$15,926.89$$

Dividing this number by 30 *gives the average daily balance*

$$ADB = \frac{SUM}{30} = \$530.90.$$

We can use the formula for interest to figure out the interest incurred (assuming 360 *days in a year),* $t = \frac{30}{360} = \frac{1}{12}$

$$I = Prt = (530.90)(0.1999)\left(\frac{1}{12}\right) = \$8.84.$$

To find the balance at the start of the next billing cycle, we add this interest to the remaining balance at the end of the last cycle:

$$\textit{New Balance} = \underbrace{\$507.43}_{\textit{Day 30 balance}} + \underbrace{\$8.84}_{\textit{Interest}} = \$516.27$$

Example 10. *A credit card has an annual interest rate of* 19.99% *and interest is calculated using the average daily balance method. If the starting balance of a* 28*-day billing cycle is* \$696.21 *and purchases of* \$25.59, \$19.95, *and* \$97.26 *are posted on days* 6, 13, *and* 25, *respectively, and a payment of* \$140 *is credited on day* 8, *what will be the balance on the card at the start of the next billing cycle?*

Solution. \$648.14